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Structure Theory for Noncommutative Jordan H^*-Algebras

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INTRODUCTION

H^*-Algebras were introduced and studied by Ambrose [4] in the associative case, and the theory has been extended to such particular classes of nonassociative algebras as Lie [15, 16], Jordan [17, 18] and alternative [13]. In all these cases the heart of the matter is showing that every H^*-algebra (in the given class) with zero annihilator is the closure of the orthogonal sum of its minimal closed ideals (each of which is a topologically simple H^*-algebra) and then listing all the topologically simple H^*-algebras in the class. In this paper we prove that every nonassociative H^*-algebra with zero annihilator is the closure of the orthogonal sum of its minimal closed ideals, we give a classification theorem for topologically simple noncommutative Jordan H^*-algebras, and we determine the topologically simple Jordan H^*-algebras. The first result is the cornerstone for a development of the theory of general nonassociative H^*-algebras (see [7]), while the last ones conclude in some sense the theory of noncommutative Jordan H^*-algebras.

1. NONASSOCIATIVE H^*-ALGEBRAS

A part of the proof of our first result can be given in the context of normed nonassociative algebras. So we recall that a normed nonassociative

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algebra V is a real or complex algebra with a norm for which the product of V is continuous. For a nonempty subset S of V we define the *left annihilator* $\mathrm{Lann}(S)$ and *right annihilator* $\mathrm{Rann}(S)$ of S in V by $\mathrm{Lann}(S) = \{x \in V: xS = 0\}$ and $\mathrm{Rann}(S) = \{x \in V: Sx = 0\}$. The *annihilator* of S in V is the set $\mathrm{Ann}(S) = \mathrm{Lann}(S) \cap \mathrm{Rann}(S)$. S^2 will denote the linear span of all $s_1 s_2$ with s_i in S. We use $M \lhd V$ to denote M is a closed ideal in V and $\bar{I}(S)$ to denote the closed ideal generated by the nonempty subset S of V. We recall that M is a trivial ideal of V if $M^2 = 0$ and V is said to be *semiprime* if 0 is its only trivial ideal (equivalently, its only closed trivial ideal). V is called *topologically simple* if $V^2 \neq 0$ and 0 and V are the only closed ideals of V. V is called *prime* if it has no nonzero orthogonal ideals (i.e., $M_1 M_2 = 0$ for $M_1, M_2 \lhd V$ implies either $M_1 = 0$ or $M_2 = 0$).

The following two concepts are crucial in our proof.

DEFINITION 1. The normed algebra V *is split by closed ideals* (or *splits*, for short) if $M \lhd V$ implies $V = M \oplus M'$ for some closed ideal M' (so $MM' = M'M = 0$), thus the lattice of closed ideals is complemented.

DEFINITION 2. A nonzero element x in the normed algebra V is *extreme* if for any closed ideal M either $x \in M$ or $x \in \mathrm{Ann}(M)$.

PROPOSITION 1. *Assume that the normed algebra V is split by closed ideals. Then*

(i) *closed ideals inherit splitting,*

(ii) *for $M \lhd V$ we have $M^2 = 0$ iff $M \subset \mathrm{Ann}(V)$, so V is semiprime iff* $\mathrm{Ann}(V) = 0$.

If in addition V is semiprime, then

(iii) $M' = \mathrm{Ann}(M)$ *for every $M \lhd V$,*

(iv) *every closed ideal is a semiprime algebra,*

(v) *for $M \lhd V$ we have that M is a nonzero prime algebra iff M is a topologically simple algebra, iff M is a minimal closed ideal in V,*

(vi) *an element x in V is extreme iff $\bar{I}(x)$ is a minimal closed ideal,*

(vii) *if V satisfies some hypothesis which is* (a) *inherited by nonzero closed ideals, and* (b) *implies the existence of minimal closed ideals* (*e.g., guarantees the existence of extreme elements*) *then $V = \overline{\bigoplus M_\alpha}$ is the closure of the direct sum of its minimal closed ideals M_α* (*which are precisely the topologically simple closed ideals of V*).

Proof. All but (vi), (vii) are immediate consequences of the definitions. For (vi), if x is extreme and $M \subset \bar{I}(x)$ a closed ideal then by extremeness either $x \in M$ (whence $\bar{I}(x) = M$) or $x \in \mathrm{Ann}(M)$ (whence $M \subset M \cap \bar{I}(x) \subset$

$M \cap \operatorname{Ann}(M)=0$ by (iii)), so $M=\bar{I}(x)$ or 0 and $\bar{I}(x)$ is minimal; conversely if $\bar{I}(x)$ is minimal and $M \lhd V$ then by (iii) $V=M \oplus M'$, $x=m+m'$ for $M'=\operatorname{Ann}(M)$, so $\bar{I}(x) \supset \bar{I}(xM)=\bar{I}(mM)$ implies by minimality either $\bar{I}(x)=\bar{I}(mM)$ (whence $x \in \bar{I}(mM) \subset M$) or $\bar{I}(mM)=0$ (whence $mM=0$, $mV=0$, $m \in \operatorname{Lann}(V)$). Similarly either $x \in M$ or $m \in \operatorname{Rann}(V)$. Thus either $x \in M$ or else $m \in \operatorname{Ann}(V)=0$. In the last case $x=m' \in M'=\operatorname{Ann}(M)$, so x is extreme. For (vii), write $M:=\overline{\bigoplus M_\alpha}$ and assume $M \neq V$, then $M' \neq 0$ so by our assumptions there is a minimal closed ideal of M' (say M_β) which is clearly a minimal closed ideal of V; thus $M_\beta \subset M \cap M'=0$; a contradiction.

DEFINITION 3. A *semi-H^*-algebra* is a complex nonassociative algebra V whose vector space is a Hilbert space satisfying $(xy \mid z)=(x \mid zy^*)=(y \mid x^*z)$ $(x, y, z \in V)$ for a given mapping $x \mapsto x^*$ from V to V (the involution of V) such that $(x+y)^*=x^*+y^*$, $(\lambda x)^*=\bar{\lambda}x^*$, $(x^*)^*=x$ for all x, y in V and λ in $\mathbb{C}$. If in addition the involution satisfies $(xy)^*=y^*x^*$ then V is called an *H^*-algebra.* It is not difficult to build a 2-dimensional commutative simple semi-H^*-algebra which is not an H^*-algebra.

PROPOSITION 2. *Let V be a semi-H^*-algebra. Then*

(i) *V is a normed algebra,*

(ii) *V is split by closed ideals: $M \lhd V \Rightarrow V=M \oplus M'$ for $M'=M^\perp$,*

(iii) $\operatorname{Lann}(M)=\operatorname{Rann}(M)=\operatorname{Ann}(M)=\overline{M^{2\perp}}$ *for all $M \lhd V$ with $M^*=M$.*

If in addition V is semiprime, then

(iv) $\operatorname{Lann}(V)=\operatorname{Rann}(V)=0$,

(v) *all closed ideals are $*$-invariant: $M^*=M$ for $M \lhd V$,*

(vi) *all closed ideals are semiprime semi-H^*-algebras,*

(vii) $\operatorname{Lann}(M)=\operatorname{Rann}(M)=\operatorname{Ann}(M)=M'=M^\perp$ *for all $M \lhd V$,*

(viii) *$M=\overline{M^2}$ for all $M \lhd V$,*

(ix) *$*$ is continuous, and is isometric if and only if V is an H^*-algebra.*

(x) *$D(M) \subset M$ for all $M \lhd V$ and every continuous derivation D of V.*

Proof. (i) Let x in V; then since the operators L_x and R_x of left and right multiplication by x have adjoint operators (L_{x^*} and R_{x^*}, respectively), it follows from the closed graph theorem that they are continuous (see [14, Theorem 5.1]); now the product of V is continuous by [14, Theorem 2.17]. (ii) $V=M \oplus M^\perp$ as closed subspaces and $M^\perp$ is clearly an ideal. (iii) $z \in \overline{M^{2\perp}} \Leftrightarrow z \in M^{2\perp} \Leftrightarrow (z \mid M^2)=0 \Leftrightarrow (zM^* \mid M)=0 \Leftrightarrow$

$(zM \mid M)=0 \Leftrightarrow (zM \mid V)=0$ (by (ii)) $\Leftrightarrow zM=0 \Leftrightarrow z \in \mathrm{Lann}(M)$, dually $\Leftrightarrow$ $z \in \mathrm{Rann}(M)$. (iv) $\mathrm{Lann}(V)=\mathrm{Rann}(V)=\mathrm{Ann}(V)=0$ by (iii). (v) $M^* \subset M$ since if $m^*=n+n' \in M \oplus M'$ then $(n'V \mid V)=(n'M' \mid V)=(m^*M' \mid V)=(M' \mid mV) \subset (M' \mid M)=0$ by (ii), so $n'V=0$, dually $Vn'=0$, $n' \in \mathrm{Ann}(V)=0$ by semiprimeness. Whenever $M^* \subset M$ for any set M we have $M=(M^*)^* \subset M^*$, so $M^*=M$. (vi) follows from (v) and Proposition 1(iv). (vii) $\mathrm{Ann}(M)=M'=M^\perp$ by (iii) and Proposition 1(iii). (viii) The closed subspaces agree since they have the same orthogonal complement: $\overline{M^2}^\perp=\mathrm{Ann}(M)=M^\perp$ by (iii) and (vii). (ix) Let $\{x_n\}$ be a sequence of elements of V such that $\lim_{n\to\infty} x_n=0$ and $\lim_{n\to\infty} x_n^*=x'$. Since the L_{x_n} operational adjoint is $L_{x_n^*}$, $\{L_{x_n^*}\}\to 0$. On the other hand, $\{L_{x_n^*}\}\to L_{x'}$. Since $\mathrm{Lann}(V)=0$ by (iv) we have $x'=0$. The closed graph theorem gives the continuity of the involution. Since $((xy)^*-y^*x^* \mid z^*)=((xy)^* \mid z^*)-(x^* \mid yz^*)=((xy)^* \mid z^*)-(x^*z \mid y)=((xy)^* \mid z^*)-(z \mid xy)$ we have $(xy)^*-y^*x^*=0$ for all $x, y \Leftrightarrow (u^* \mid z^*)=(z \mid u)$ for all $u \in V^2 \Leftrightarrow (u^* \mid z^*)=(z \mid u)$ for all $u \in \overline{V^2}=V$ (by (viii)), thus $*$ is an algebra involution (V is an H^*-algebra) $\Leftrightarrow *$ is isometric. (x) Always $D(M^2) \subset D(M)\,M+MD(M) \subset M$ so, by (viii), $D(M)=D(\overline{M^2}) \subset M$ since D is continuous and M closed.

PROPOSITION 3. *Each nonzero semiprime semi-H^*-algebra V has extreme elements. More concretely, if $\hat{V}$ denotes V under the norm $|x|=\|L_x\|$, then for each extreme point f in the closed unit ball B of the dual space $\hat{V}'$ of the normed space $\hat{V}$ there is an extreme element v of V such that $f(x)=(x \mid v)$ for all x in V.*

Proof. By Proposition 2(i) there is a positive number k such that $\|xy\| \leqslant k\,\|x\|\,\|y\|$ $(x, y \in V)$, that is: $|x|=\|L_x\| \leqslant k\,\|x\|$ $(x \in V)$, so each f in $\hat{V}'$ is contained in V', $|f(x)| \leqslant |f|\,|x| \leqslant k\,|f|\,\|x\|$, and therefore by the Riesz representation theorem has the form $f(x)=(x \mid v)$ for some vector v in V. If f is an extreme point in B we claim v is an extreme element in V: if $M \lhd V$ then by Proposition 2(ii), $V=M \oplus \mathrm{Ann}(M)=M_1 \oplus M_2$, and we must show $v \in M_1$ or M_2, i.e., $v \in M_2^\perp$ or $M_1^\perp$ (by Proposition 2(ii)), i.e., $f(M_2)=0$ or $f(M_1)=0$, i.e., $f \in M_2^0$ or M_1^0 (those continuous functionals on $\hat{V}$ vanishing on M_2 or M_1 resp.). But since $V=M_1 \oplus M_2$ is an L_2-direct sum of Hilbert spaces and for m_i in M_i we have $L_{m_i}(M_i) \subset M_i$, $L_{m_i}(M_j)=0$ for $i \neq j$, a routine calculation shows $\|L_{m_1+m_2}\|=\mathrm{Max}\{\|L_{m_1}\|, \|L_{m_2}\|\}$ (taking $\|L_{m_i}\|$ on M_i or all V), so $|m_1+m_2|=\mathrm{Max}\{|m_1|, |m_2|\}$ shows $\hat{V}=M_1 \oplus M_2$ is an M-direct sum of normed spaces; hence as is well known $\hat{V}'=M_1^0 \oplus M_2^0$ is an L-direct sum $(|f_1+f_2|=|f_1|+|f_2|)$, and therefore by [2, Proposition 1.15], f extreme in B implies $f \in M_1^0$ or M_2^0 as claimed. The unit ball B is compact in the weak* topology by the Banach–Alaoglu theorem, hence has nonzero extreme points by the Krein–Milman theorem. Therefore V has extreme elements.

THEOREM 1. *A nonzero semi-H^*-algebra V is semiprime iff $V=\overline{\perp M_\alpha}$ is the closure of an orthogonal sum of topologically simple semi-H^*-algebra M_α (namely its minimal closed ideals).*

Proof. "Only if" follows from Proposition 1(vii) since the property of being nonzero semiprime semi-H^*-algebra is inherited by nonzero closed ideals (Proposition 2(vi)) and implies the existence of extreme elements by Proposition 3; the M_α are clearly orthogonal since $\alpha \neq \beta$ implies $M_\alpha M_\beta = M_\beta M_\alpha = 0 \Leftrightarrow M_\alpha \subset \operatorname{Ann}(M_\beta) = M_\beta^\perp$ by Propositions 1(iii) and 2(ii). For the converse: if $V=\overline{\perp M_\alpha}$ then for z in $\operatorname{Ann}(V)$ we have $(z \mid \overline{M_\alpha^2})=0$; but, since M_α is a topologically simple (so semiprime) semi-H^*-algebra $\overline{M_\alpha^2}=M_\alpha$ (Propositions 2(viii), with $M=V=M_\alpha$); thus $(z \mid M_\alpha)=0$, $(z \mid V)=0$, $z=0$. Therefore $\operatorname{Ann}(V)=0$ and V is semiprime by Propositions 2(ii) and 1(ii).

2. NONCOMMUTATIVE JORDAN H^*-ALGEBRAS

In this section we give a classification theorem for topologically simple noncommutative Jordan H^*-algebras the proof of which uses results [17] on commutative algebras and results [3] about noncommutative Jordan algebras.

THEOREM 2. *If V is a topologically simple noncommutative Jordan H^*-algebra then either*

(1) *V is anticommutative (V^+ is trivial).*

(2) *V is commutative ($V=V^+$).*

(3) *V is quadratic: $*$ the involution of V is isometric, $V=\mathbb{C}e \perp W$ with e the unit of V and $xy=((x \mid y^*)/\|e\|^2)\, e + x \wedge y$ ($x, y \in W$), where $(W, \wedge)$ is a $*$-invariant anticommutative algebra with dimension greater than 1.*

(4) *V is quasi-associative: $V=D^{(\lambda)}$ with $\lambda \in \mathbb{R}-\{\frac{1}{2}\}$ and D a topologically simple associative noncommutative H^*-algebra with the same inner product and involution as V.*

Proof. Since $\operatorname{Ann}(V^+)$ is a closed ideal of V^+ invariant under all derivations of V^+, by flexibility it is invariant under all D_x: $y \mapsto [x, y]$, so $\operatorname{Ann}(V^+)$ is an ideal in V. If $\operatorname{Ann}(V^+)=V^+$ we have Case 1, otherwise by topological simplicity we have $\operatorname{Ann}(V^+)=0$. In the latter case V^+ remains topologically simple: if $M \lhd V^+$ then M is invariant under all D_x by Proposition 1(ii) and Proposition 2(x) since V^+ remains a semi-H^*-algebra; hence M is an ideal in V.

So assume V and V^+ are both topologically simple. Hence V^+ is a Jordan H^*-algebra with $\operatorname{Ann}(V^+)=0$. By [17, Proposition 4.3, 3.4], V^+ has a maximal family of completely primitive pairwise orthogonal projections $\{e_i\}_{i \in \mathscr{A}}$ (that is, nonzero pairwise orthogonal idempotents that satisfies $e_i^* = e_i$, $V_1^+(e_i) = \mathbb{C}e_i$) and V is the closure

$$V = \overline{\perp V_{ij}^+}$$

of the orthogonal direct sum of all different Peirce subspaces V_{ij}^+ which are defined by the equations

$$V_{ii}^+ = \{x \in V : x \cdot e_i = x\}$$

$$V_{ij}^+ = \{x \in V : x \cdot e_i = \tfrac{1}{2}x = x \cdot e_j\} \qquad (i \neq j)$$

(cf. [1, pp. 559–562]). Furthermore the e_i are $*$-strongly connected (that is, there exist $u_{ij} \in V_{ij}^+$ with $u_{ij}^* = u_{ij}$ and $u_{ij}^2 = e_i + e_j$, for $i \neq j$). In fact: By the well-known properties of the Peirce decomposition we have

$$x_{ij}^2 = Q_{ij}(x_{ij})(e_i + e_j) \qquad (x_{ij} \in V_{ij}^+,\ i \neq j)$$

for a quadratic form Q_{ij} on $V_{ij}^+ = S_{ij} + (-1)^{1/2} S_{ij}$ (S_{ij} the real subspace of the self-adjoint elements of V_{ij}^+). By [17, p. 299], $S_{ij} \neq 0$. If $x_{ij} \in S_{ij}$ then $\|x_{ij}\|^2 = (x_{ij} \mid x_{ij} \cdot (e_i + e_j)) = (x_{ij}^2 \mid e_i + e_j) = Q_{ij}(x_{ij})\ \|e_i + e_j\|^2$. This shows $Q_{ij}(x_{ij}) > 0$ for nonzero self-adjoint x_{ij} of V_{ij}^+; in particular we can replace x_{ij} by $u_{ij} = Q_{ij}(x_{ij})^{-1/2} x_{ij}$ to get $u_{ij}^* = u_{ij}$, $Q_{ij}(u_{ij}) = 1$.

If $\mathscr{A} = \{1\}$ then $V = \mathbb{C}e_1$ is commutative.

If $\mathscr{A} = \{1, 2\}$ then $V^+ = \mathbb{C}e_1 \perp \mathbb{C}e_2 \perp V_{12}^+$ has V^+ (hence V) quadratic (cf. [9, p. 202]). Since $V_{12}^+ \neq 0$ we have dimension $V > 2$. By [11, p. 203], $V = \mathbb{C}e \oplus W$ with product $xy = f(x, y)\, e + x \wedge y$ for a symmetric bilinear form f, $W = \{x \in V : f(x, e) = f(e, x) = 0\}$ and product $\wedge$ with $e \wedge w = w \wedge e = w$, $e \wedge e = 0$, $w \wedge w' = -w' \wedge w \in W$ for any $w, w' \in W$. Since $W = \mathbb{C}(e_1 - e_2) \perp V_{12}^+$ and $(e_1 - e_2 \mid e) = (e_1 - e_2 \mid u_{12}^2) = ((e_1 - e_2) \cdot u_{12} \mid u_{12}) = 0$ we have $V = \mathbb{C}e \perp W$. For self-adjoint u in V we have $\|u\|^2 = (u \mid u) = (u^2 \mid e) = f(u, u)\ \|e\|^2$, so $(u \mid v) = f(u, v)\ \|e\|^2$ for u, v in the real subspace S of the self-adjoint elements of V. Hence $f(x, y) = (x \mid y^*)/\|e\|^2$ for any $x, y \in V = S + (-1)^{1/2} S$, and the product is $xy = ((x \mid y^*)/\|e\|^2)\, e + x \wedge y$. Here $*$ is isometric on V since $\|x\|^2 = (x \mid x) = (xe \mid x) = (e \mid x^*x) = (ex^* \mid x^*) = \|x^*\|^2$, and $W = e^\perp$ with $e^* = e$ shows $W^* = W$. Thus V is as in Case 3.

Asume $|\mathscr{A}| \geqslant 3$. Then

$$V = \overline{\bigcup_{F \subset \mathscr{A}, 3 \leqslant |F| < \infty} V_F}, \qquad V_F = \perp_{i,j \in F} V_{ij} = V_1(e_F), \qquad e_F = \sum_{i \in F} e_i,$$

where the Peirce subspaces $V_{ij} = V_{ij}^+$ coincide in V and V^+ and each V_F is a unital noncommutative Jordan H^*-algebra with V_F^+ an algebraically simple Jordan algebra (by [17, Theorem 4.6]). Notice also that the simple unital H^*-algebras (in particular the V_F) are central. In fact, its center is a field, which by the Gelfand–Mazur theorem must be $\mathbb{C}e$. At this point we can apply to each V_F the following theorem of Alvermann [3, Theorem 4.5, 4.6]: if K is a field of characteristic different from 2 and J a central noncommutative Jordan algebra over K with $n \geqslant 3$ strongly connected supplementary orthogonal idempotents then either

(a) $[J, [J, J]] = 0$.

(b) J is (commutative) Jordan.

(c) J is quasi-associative: there is a quadratic extension L of K with J_L split quasi-associative ($J_L = D^{(\lambda)}$ for D associative).

If each V_F is as in Case (a) then $[V, [V, V]] = 0$. We first rule out this case by showing V is commutative. Since $V = S + (-1)^{1/2} S$ for S self-adjoint elements it suffices to show $[S, V] = 0$, i.e., $L_x = R_x$ for $x = x^*$; but $[x, [x, V]] = 0$ is equivalent to $0 = (L_x - R_x)^2 = (L_{x^*} - R_{x^*})(L_x - R_x) = (L_x - R_x)^*(L_x - R_x)$, and $T^*T = 0$ forces $T = 0$ for any continuous operator on Hilbert space, thus $L_x = R_x$.

So each V_F is either commutative or quasi-associative. If all V_F are commutative so is V and we have Case 2. If some V_{F_0} is not commutative neither is any V_F for $F \supset F_0$, so all such V_F are quasi-associative and there exists an associative noncommutative algebra D_F and an $\lambda_F \in \mathbb{C} - \{\frac{1}{2}\}$ such that $V_F = D_F^{(\lambda_F)}$. By [12, Proposition 2.5] the above λ_F are the same complex number λ. Thus $V = \overline{\bigcup_{\mathscr{A} \supset F \supset F_0} V_F} = \overline{\bigcup D_F^{(\lambda)}} = D^{(\lambda)}$ for $D = \bigcup_{\mathscr{A} \supset F \supset F_0} D_F$ an associative noncommutative algebra, being $D = V^{(\mu)}$, where $\mu = \lambda/(2\lambda - 1)$. Now we claim that λ is real. Assume $\lambda \notin \mathbb{R}$. By Proposition 1(ii) V is semiprime and so is D. By [12, Proposition 2.2], $\mathrm{Re}(\lambda) = \frac{1}{2}$ and $*$ is multiplicative in D (that is, the product $\underset{(\mu)}{\cdot}$ of D satisfies $(x \underset{(\mu)}{\cdot} y)^* = x^* \underset{(\mu)}{\cdot} y^*$). An elementary computation using $\bar{\mu} = 1 - \mu$ shows

$$(x \underset{(\mu)}{\cdot} y \mid z) = (x \mid y^* \underset{(\mu)}{\cdot} z) = (y \mid z \underset{(\mu)}{\cdot} x^*) \tag{1}$$

for any $x, y, z \in D$. Let L_x, R_x be the linear maps $L_x\colon y \mapsto x \underset{(\mu)}{\cdot} y$ and $R_x\colon y \mapsto y \underset{(\mu)}{\cdot} x$. By (1) and the associativity of D, L_x is a normal operator. So its spectral radius $r(L_x)$ satisfies $r(L_x) = \|L_x\|$. Similarly $r(R_x) = \|R_x\|$. But it is known that an associative normed algebra in which the spectral radius and the norm agree must be commutative (see [5, Corollary 15.7 and the comment below]). Therefore the sets $\{L_x\colon x \in D\}$ and $\{R_x\colon x \in D\}$ are commutative algebras. Thus we have $L_{[x,y]} = [L_x, L_y] = 0$ and $R_{[x,y]} = [R_x, R_y] = 0$ for any $x, y \in D$. Hence $[x, y] \in \mathrm{Ann}(D) = \mathrm{Ann}(V) = 0$ and D is commutative; a contradiction. This contradiction gives λ real. From $D = V^{(\mu)}$ we obtain that D is a topologically simple H^*-algebra and we have Case 4.

Remark 1. Let V be a topologically simple noncommutative Jordan semi-H^*-algebra. As in the proof of Theorem 2 either V is anticommutative or V^+ is a topologically simple Jordan semi-H^*-algebra. Following the ideas of [10] we can prove that every Jordan semi-H^*-algebra with zero annihilator is an H^*-algebra. Asume V^+ is a topologically simple Jordan semi-H^*-algebra. Therefore V^+ is an H^*-algebra, so by Proposition 2(ix), $*$ is isometric. Proposition 2(ix) now gives that V is an H^*-algebra. Therefore our Theorem 2 is also valid if we replace "H^*-algebra" by "semi-H^*-algebra."

We will examine the commutative case in more detail in the next section.

3. JORDAN H^*-ALGEBRAS

The Jacobson coordinatization theorem plays a fundamental role in the classification of the nondegenerate simple Jordan algebras with unity satisfying the minimum conditions for quadratic ideals. In this section we prove a similar coordinatization theorem for Jordan H^*-algebras and use it to classify all the topologically simple Jordan H^*-algebras. A partial result was given by C. Viola Devapakkiam and P. S. Rema [18].

We recall that a family $\{e_i\}_{i\in\mathscr{A}}$ of nonzero pairwise orthogonal projections of a Jordan H^*-algebra V is called maximal if it is not properly contained in any other. Let V and W be H^*-algebras. The homomorphism $f\colon V\to W$ is called a *$*$-homomorphism* if $f(x^*)=f(x)^*$ for any $x\in V$. We call a bijective $*$-homomorphism a *$*$-isomorphism*, and a $*$-isomorphism from V to V a *$*$-automorphism.*

PROPOSITION 4. *Let V be a Jordan H^*-algebra with zero annihilator and let $\{e_i\}_{i\in\mathscr{A}}$ be a maximal family of nonzero pairwise orthogonal projections whose cardinal is greater than or equal to 3. Let ω_1 be an element of $\mathscr{A}$ such that any e_j is $*$-strongly connected to e_{ω_1} via $u_{\omega_1 j}$. For i, j in $\mathscr{A}$ such that $i\neq j$ and both are different from ω_1, we define*

$$u_{ij}=2u_{\omega_1 i}u_{\omega_1 j}.$$

If now we assume that i, j are different arbitrary elements of $\mathscr{A}$ there exists a unique continuous linear map $T_{ij}\colon V\to V$ such that

(1) $T_{ij}(x)=U_{u_{ij}}(x)$ *if* $x\in V_{ii}\perp V_{ij}\perp V_{jj}$.

(2) $T_{ij}(x)=x$ *if* $x\in V_{kl}$ *and* $k, l\neq i, j$.

(3) $T_{ij}(x)=2R_{u_{ij}}(x)$ *if* $x\in V_{ik}\perp V_{jk}$ *and* $k\neq i, j$.

Every T_{ij} is an isometric $*$-automorphism whose square is the identity and the map $(\omega_1 j)\mapsto T_{\omega_1 j}$ can be extended in a unique way to a monomorphism

$\varphi: \pi \mapsto T_\pi$ from the group $\mathscr{I}_{\mathrm{fin}}(\mathscr{A})$ of permutations of $\mathscr{A}$ fixing all but a finite number of elements to the group of the isometric $*$-automorphisms of V. This monomorphism satisfies $\varphi((ij)) = T_{ij}$. Moreover $\varphi(\pi)(x) = \varphi(\pi')(x)$ if $x \in V_{kl}$ and $\pi(k) = \pi'(k)$, $\pi(l) = \pi'(l)$ ($k = l$ allowed).

Proof. The unicity of every T_{ij} it follows from $V = \bar{V}_0$ where $V_0 = \bigcup_{F \subset \mathscr{A}, |F| < \infty} V_F$ and $V_F = V_1(\sum_{i \in F} e_i)$ (see [17, Theorem 3.5]). Applying the usual Jacobson Coordinatization Theorem to the unital subalgebras V_F for finite subsets $F \subset \mathscr{A}$ we obtain a monomorphism φ_0 from $\mathscr{I}_{\mathrm{fin}}(\mathscr{A})$ to the algebra automorphisms of V_0 such that $\varphi_0((ij))$ and T_{ij} agree on V_0. Let $\varphi(\pi): V \to V$ be the continuous extension of $\varphi_0(\pi)$. So $\varphi(\pi)$ is an automorphism of V whose inverse is $\varphi(\pi^{-1})$ and the map $\varphi: \pi \mapsto T_\pi$ is a monomorphism of groups. Since $T_{ij} = U_{t_{ij}}$ ($t_{ij} = 1 - e_1 - e_j + u_{ij}$, 1 the unit in the unital hull of V and $U_y = 2R_y^2 - R_{y^2}$) we see $T_{ij}^* = T_{ij} = T_{ij}^{-1}$, so $T_\pi^* = T_\pi^{-1}$ for all $\pi \in \mathscr{I}_{\mathrm{fin}}(\mathscr{A})$. Thus $\|T_\pi(x)\|^2 = (T_\pi(x) \mid T_\pi(x)) = (T_\pi^* T_\pi(x) \mid x) = (x \mid x) = \|x\|^2$, so all T_π are isometric. Furthermore T_π respect the involution since the T_{ij} do. In fact, $T_{ij}(x)^* = U_{t_{ij}}(x)^* = U_{t_{ij}}(x^*) = T_{ij}(x^*)$. We get by [9, p. 134, Lemma 2] that if $\pi(k) = \pi'(k)$, $\pi(l) = \pi'(l)$ then $T_\pi(x) = T_{\pi'}(x)$ for $x \in V_{kl}$.

LEMMA 1. *Let V, $\{e_i\}_{i \in \mathscr{A}}$ and u_{ij} be as in Proposition 4. Then $\|R_{u_{ij}}(x)\|^2 = \frac{1}{2}\|x\|^2$ for $x \in V_{ii}$. Moreover $\|u_{ij}\|^2 = 2\varepsilon$, where all $\|e_k\|^2 = \varepsilon$.*

Proof. By [9, p. 136] we have $2R_{e_i}R_{u_{ij}}^2(x) = x$ for any $x \in V_{ii}$. The Peirce decomposition properties give $R_{u_{ij}}^2(x) \in V_{ii} \perp V_{jj}$. So we have

$$\|R_{u_{ij}}(x)\|^2 = (R_{u_{ij}}(x) \mid R_{u_{ij}}(x)) = (x \mid R_{u_{ij}}^2(x)) = (x \mid R_{e_i}R_{u_{ij}}^2(x)) = \tfrac{1}{2}\|x\|^2.$$

In particular $\|u_{ij}\|^2 = 2\|e_i\|^2 = 2\|e_j\|^2$, so all $\|e_i\|^2 = \varepsilon$. ▮

Let $\mathscr{A}$ be a non-vacuous set and let D be a semi-H^*-algebra with continuous involution $a \mapsto a^\square$. Then the set $\mathscr{M}_{\mathscr{A}}(D)$ of matrices $A = (a_{ij})$ that have entries in D and satisfy $\sum \|a_{ij}\|^2 < \infty$ becomes a semi-H^*-algebra under the inner product $(A \mid B) = \frac{1}{2}\sum_{(i,j) \in \mathscr{A} \times \mathscr{A}} (a_{ij} \mid b_{ij})$ (so the norm is determined by $\|A\|^2 = \frac{1}{2}\sum_{(i,j) \in \mathscr{A} \times \mathscr{A}} \|a_{ij}\|^2$), the algebra product $AB = (c_{ij})$ for $c_{ij} = \sum_{k \in \mathscr{A}} a_{ik}b_{kj}$ and the involution $A \mapsto A^* = (a_{ji}^\square)$ (A^* the transpose matrix of that obtained by applying $\square$ to each entry of A). $\mathscr{M}_{\mathscr{A}}(D)$ is an H^*-algebra if D is so. We shall call it *the semi-H^*-algebra (H^*-algebra) of the $\mathscr{A} \times \mathscr{A}$-matrices with entries in D.* If D carries an involutive continuous antiautomorphism s such that $s(a^\square) = [s(a)]^\square$ for any $a \in D$, then $S(A) = (s(a_{ji}))$ is an involutive continuous antiautomorphism of $\mathscr{M}_{\mathscr{A}}(D)$ that satisfies $S(A)^* = S(A^*)$, so the set $\mathscr{H}_{\mathscr{A}}(D, s) = \{A \in \mathscr{M}_{\mathscr{A}}(D) : S(A) = A\}$ is a closed self-adjoint subalgebra of $\mathscr{M}_{\mathscr{A}}(D)^+$ which is a semi-H^*-algebra under the restrictions of the inner product and the involution of $\mathscr{M}_{\mathscr{A}}(D)$ (thus $A^* = (a_{ij}^*)$ for $a^* = s(a)^\square = s(a^\square)$). Moreover $\mathscr{H}_{\mathscr{A}}(D, s)$ is an H^*-

algebra if D is so. We shall call it *the semi-H^*-algebra (H^*-algebra) of the s-symmetric $\mathscr{A}\times\mathscr{A}$-matrices with entries in D.* Assume D has unity; if $i, j \in \mathscr{A}$ we denote by E_{ij} the matrix of $\mathscr{M}_{\mathscr{A}}(D)$ that has the unity of D in the $(i; j)$-entry and zero in the remaining entries. If $a \in D$ we write $a[ij] = aE_{ij} + s(a)\,E_{ji}$. Every A in $\mathscr{H}_{\mathscr{A}}(D, s)$ has the form $A = \sum_{i \leqslant j} a_{ij}[ij]$, where $\sum_{i \leqslant j} \|a_{ij}\|^2 < \infty$ and $\leqslant$ denotes a well order on $\mathscr{A}$.

THEOREM 3 (Coordinatization in Jordan H^*-Algebras). *Let V be a Jordan H^*-algebra with zero annihilator, $\{e_i\}_{i\in\mathscr{A}}$ a maximal family of nonzero pairwise orthogonal projections whose cardinal is greater than or equal to 3. Let ω_1, ω_2, ω_3 be three different elements of $\mathscr{A}$. Assume that for all $i \in \mathscr{A}$, e_i is $*$-strongly connected to e_{ω_1} via $u_{\omega_1 i}$. Then*

(1) *$D = V_{\omega_1\omega_2}$ is a unital alternative algebra over $\mathbb{C}$ whose product is $x\nabla y = 2T_{\omega_2\omega_3}(x)\,T_{\omega_1\omega_3}(y)$, whose unit is $u = u_{\omega_1\omega_2}$ and where the map $s\colon D \to D$ given by $s(x) = T_{\omega_1\omega_2}(x)$ is an involutive continuous antiautomorphism. This algebra is associative if* $\operatorname{card}(\mathscr{A}) > 3$.

(2) *$D = V_{\omega_1\omega_2}$ is an H^*-algebra under the restriction of the inner product of V and the involution defined by $x^{\sqcup} = s(x^*)$.*

(3) *There is an isometric $*$-isomorphism from V to the H^*-algebra of the s-symmetric $\mathscr{A}\times\mathscr{A}$-matrices with entries in D, $\mathscr{H}_{\mathscr{A}}(D, s)$.*

Proof. The algebraic properties (1) of D follow from the Jacobson strong coordinatization theorem [9, p. 133] applied to V_F for $F = \{\omega_1, ..., \omega_n\} \subset \mathscr{A}$ ($n = 3$ or 4). For (2), note by commutativity of V $*$ is an involutive $\mathbb{C}$-antilinear automorphism of V with $e_i^* = e_i$ which commutes with all T_{ij}, so it induces a $\mathbb{C}$-antilinear involutory automorphism of $D = V_{\omega_1\omega_2}$ commuting with $s = T_{\omega_1\omega_2}$, and $x^{\sqcup} = s(x^*)$ is an involution on D. We have

$$\begin{aligned}(x\nabla y \mid z) &= 2(T_{\omega_2\omega_3}(x)\,T_{\omega_1\omega_3}(y) \mid z) = 2(T_{\omega_2\omega_3}(x) \mid zT_{\omega_1\omega_3}(y^*)) \\ &= 2(x \mid T_{\omega_2\omega_3}(z)\,T_{\omega_2\omega_3}T_{\omega_1\omega_3}(y^*)), \end{aligned} \tag{2}$$

after making use of Proposition 4. On the other hand,

$$(x \mid z\nabla y^{\sqcup}) = 2(x \mid T_{\omega_2\omega_3}(z)\,T_{\omega_1\omega_3}T_{\omega_1\omega_2}(y^*)). \tag{3}$$

From (2), (3), Proposition 4 and $(\omega_2\omega_3)(\omega_1\omega_3) = (\omega_1\omega_3)(\omega_1\omega_2)$ we obtain $(x\nabla y \mid z) = (x \mid z\nabla y^{\sqcup})$ for $x, y, z \in D$. Similarly it can be proved that $(x\nabla y \mid z) = (y \mid x^{\sqcup}\nabla z)$.

We know from the usual coordinatization theorem that there is an algebraic isomorphism $\eta\colon V_0 = \perp V_{ij} \to \mathscr{H}_{\mathscr{A}}(D, s)$ by $\eta(x) = \sum_{i\leqslant j} \eta_{ij}(x_{ij})[ij]$

($\eta_{ij}(x_{ij}) = T_\pi(x_{ij})$ for $\pi(i) = \omega_1$, $\pi(j) = \omega_2$, $\eta_{ii}(x_{ii}) = \eta_{ij}(x_{ii} \cdot u_{ij})$ and $\leqslant$ a well order on $\mathscr{A}$). Moreover

$$\eta(x^*) = \sum_{i \leqslant j} \eta_{ij}(x_{ij}^*)[ij] = \sum_{i \leqslant j} s(\eta_{ij}(x_{ij}^*))[ji] = \eta(x)^*. \tag{4}$$

By Proposition 4, $\eta_{ij} = T_\pi$ is an isometry for $i \leqslant j$, so $2\,\|\eta_{ii}(x_{ii})\|^2 = 2\,\|x_{ii} \cdot u_{ij}\|^2 = \|x_{ii}\|^2$ by Lemma 1. Hence η is an isometry on V_0 because

$$\begin{aligned}\|\eta(x)\|^2 = \left\| \sum_{i \leqslant j} \eta_{ij}(x_{ij})[ij] \right\|^2 &= \frac{1}{2} \sum_{i<j} \|\eta_{ij}(x_{ij})\|^2 \\ &+ \frac{1}{2} \sum_{i<j} \|s\eta_{ij}(x_{ij})\|^2 + \frac{1}{2} \sum_i \|2\eta_{ii}(x_{ii})\|^2 \\ &= \sum_{i<j} \|\eta_{ij}(x_{ij})\|^2 + 2 \sum_i \|\eta_{ii}(x_{ii})\|^2 \\ &= \sum_{i<j} \|x_{ij}\|^2 + \sum_i \|x_{ii}\|^2 = \|x\|^2.\end{aligned}$$

There exists a unique continuous homomorphism G from $V = \bar{V}_0$ to $\mathscr{H}_{\mathscr{A}}(D, s)$ that extends η. We have

$$\|G(z)\|^2 = \|z\|^2 \tag{5}$$

for any $z \in V$. From (5) it follows that G is a monomorphism whose image is closed. This image is dense, therefore G is an isomorphism. By Proposition 2(ix) (4) and (5), G is an isometric $*$-isomorphism. ▌

Let D_0 be one of the four real composition division algebras ($\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, $\mathbb{O}$), s_0 its canonical involutive antiautomorphism and Q_0 its quadratic form. Let $D = (D_0)_{\mathbb{C}} = D_0 \otimes_{\mathbb{R}} \mathbb{C}$ the complexification of D_0. There is a unique involutive antiautomorphism s (resp. quadratic form Q, multiplicative involution $\square$) that satisfies $s(x_0 \otimes \lambda) = s_0(x_0) \otimes \lambda$ (resp. $Q(x_0 \otimes \lambda,\ y_0 \otimes \mu) = \lambda\mu Q_0(x_0, y_0)$, $(x_0 \otimes \lambda)^{\square} = s_0(x_0) \otimes \bar{\lambda}$). D becomes an alternative H^*-algebra with respect the involution $\square$ and the inner product determined by $(x_0 \otimes \lambda \mid y_0 \otimes \mu) = Q(x_0 \otimes \lambda,\ y_0 \otimes \bar{\mu})$. D is a composition algebra of quadratic form Q which we shall call the *canonical composition H^*-algebra.*

THEOREM 4. *The Jordan H^*-algebra V is topologically simple iff there exists a $*$-isomorphism multiplying all the norms by the same positive factor, from V to one of the following H^*-algebras:*

(1) $\mathbb{C}$.

(2) $\mathbb{C}e \oplus H$ *the quadratic Jordan H^*-algebra associated with a com-*

plex Hilbert space H of dimension strictly greater than 1 *with isometric involutive semilinear map* $\square$: $(\alpha e + x)(\beta e + y) = [\alpha\beta + (x \mid y^{\square})]\, e + \beta x + \alpha y$, $(\alpha e + x \mid \beta e + y) = \alpha\bar{\beta} + (x \mid y)$, $(\alpha e + x)^* = \bar{\alpha} e + x^{\square}$.

(3) *$\mathscr{H}_{\mathscr{A}}(D, s)$ the H^*-algebra of all s-symmetric $\mathscr{A} \times \mathscr{A}$-matrices with entries in D, where $\mathscr{A}$ is a set of cardinals greater than or equal to* 3 *and D is one of the four canonical composition H^*-algebras, with D different from the complex octonions if* $\operatorname{card}(\mathscr{A}) > 3$.

Proof. If V is topologically simple then $\operatorname{Ann}(V) = 0$ because $\operatorname{Ann}(V)$ is a closed ideal of V. Let $\{e_i\}_{i \in \mathscr{A}}$ be a maximal family of completely primitive pairwise orthogonal projections. If the family has only one element e then $V = V_1(e) = \mathbb{C}e$. If $\mathscr{A}$ has two elements we see as in the proof of Theorem 2 that V is as in Case 3 of Theorem 2. Since V is commutative, W has null product. If e is the unit of V, multiplying by a positive factor we can change the inner product so that $\|e\| = 1$. Then V is as in Case 2 with $H = W$ and $\square$ the map induced by $*$. Conversely all such V are simple.

We now assume $|\mathscr{A}| \geqslant 3$. As in the proof of Theorem 2 every two different elements of $\{e_i\}_{i \in \mathscr{A}}$ are $*$-strongly connected. By our Coordinatization Theorem for Jordan H^*-algebras, $D = V_{\omega_1\omega_2}$ is a unital alternative H^*-algebra with involutive continuous antiautomorphism s and there exists an isometric $*$-isomorphism from V to $\mathscr{H}_{\mathscr{A}}(D, s)$. Since $V_{ii} = \mathbb{C}e_i$ we have D a composition algebra [9, p. 205]. We denote by $Q(x)$ its quadratic form. Using the notations of Theorem 3 we have $Q(x)\,\|u\|^2 = (Q(x)\, u \mid u) = (x \nabla s(x) \mid u) = (x \mid u \nabla s(x)^{\square}) = (x \mid s(x)^{\square}) = (x \mid x^*)$, so $Q(x) = (x \mid x^*)/\|u\|^2$. We know $D = D_0 \oplus iD_0 = (D_0)_{\mathbb{C}}$ for $D_0 = \{x \in V_{\omega_1\omega_2} \colon x^* = x\}$. We have $x_0 \nabla s(x_0) = s(x_0) \nabla x_0 = Q(x_0)\, u = [(x_0 \mid x_0^*)/\|u\|^2]\, u = [\|x_0\|^2/\|u\|^2]\, u$ for $x_0 \in D_0$. Thus D_0 is one of the four positive-definite real composition division algebras. By Theorem 3 the involution of D is $x \mapsto s(x^*)$, so if we change the inner product multiplying by a convenient positive factor, then D agrees with the canonical composition H^*-algebra. Thus V is as in Case 3. Conversely it is easy to see that all such $V = \mathscr{H}_{\mathscr{A}}(D, s)$ of Case 3 are topologically simple (with nonzero ideals containing the finite matrices).

Remark 2. Let $\mathbb{K}$ be the division ring of the real, complex or quaternions. Let H be a right vector space over $\mathbb{K}$ and

$$(\ \mid\ ) \colon H \times H \to \mathbb{K}$$

a symmetric bilinear hermitian form such that $(x \mid x) > 0$ for any $x \in H - \{0\}$ (see [8, p. 74] for definition). We shall call H a prehilbertian space over $\mathbb{K}$. We define a norm $\|\ \|$ in H by the formula

$$\|x\| = (x \mid x)^{1/2}.$$

If H is complete under $\| \ \|$ we ay that H is a Hilbert space over $\mathbb{K}$. As in the classical case ($\mathbb{R}$ or $\mathbb{C}$) we may define the orthonormal complete systems, the operational adjoint of a continuous linear map of H and the Hilbert–Schmidt operators of H. The set of all the Hilbert–Schmidt operators of H that are equal to their operational adjoint is in a natural way a Jordan $\mathbb{R}$-algebra and a real Hilbert space. This algebra by complexifying gives an algebra that may be identified with the H^*-algebra $\mathscr{H}_{\mathscr{A}}(D, s)$, where D is the canonical composition H^*-algebra obtained from $\mathbb{K}$ by complexifying and $\mathscr{A}$ is a set of cardinality equal to the Hilbert dimension of H. This identification allows us to represent the Jordan H^*-algebras of (3) of Theorem 4 different from $\mathscr{H}_3(\mathbb{O}_{\mathbb{C}}, s)$ as complexified algebras of self-adjoint Hilbert–Schmidt operators on a Hilbert space over $\mathbb{K}$ of Hilbert dimension greater than or equal to 3 (for details, see [6]).

Remark 3. From Theorem 4, the above remark and [17, Proposition 4.7] the result of C. Viola and P. S. Rema on infinite-dimensional separable "special" topologically simple Jordan H^*-algebras [18] can be easily deduced. It is enough to observe that the Hilbert space H over $\mathbb{K}$ considered in Remark 2 is a real Hilbert space for the inner product $(x, y) \mapsto (x, y) + s((x \mid y))$, so every self-adjoint $\mathbb{K}$-linear operator on H is $\mathbb{R}$-linear and therefore is extended to a self-adjoint $\mathbb{C}$-linear operator on the complex Hilbert space obtained by complexifying of the real Hilbert space H.

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